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Lie derivatives of homogeneous structures of real hypersurfaces in a complex space form

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Abstract. In this paper we calculate the Lie derivatives of homogeneous structure tensors of homogeneous real hypersurfaces in nonflat complex space forms in the direction of the structure vector field. Using these results, we give two characterization theorems of a homogeneous real hypersurface of type (A) in a nonflat complex space form.

1. Introduction

Let $M_n(c)$ be an n -dimensional complex space form with constant holomorphic sectional curvature $c \neq 0$, and let \tilde{J} and g be its complex structure and Riemannian metric. Complete and simply connected complex space forms are isometric to a complex projective space $\mathbb{C}P_n$ or a complex hyperbolic space $\mathbb{C}H_n$ for $c > 0$ or $c < 0$, respectively.

Let M be a connected submanifold of $M_n(c)$ with real codimension 1. We refer to this simply as a real hypersurface below. For a local unit normal vector field ν of M , we define the structure vector field ξ of M by $\iota_*\xi = -\tilde{J}\nu$, where ι_* denotes the differential map of the immersion map ι of M into $M_n(c)$. Further, the structure tensor field ϕ and the 1-form η are defined by $\tilde{J}\iota_*X = \iota_*\phi X + g(X, \xi)\nu$, $\eta(X) = g(X, \xi)$ for a tangent vector X of M , where g denotes the induced Riemannian metric of M .

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The structure vector ξ is said to be principal if $A\xi = \alpha\xi$ is satisfied for some function α , where A is the shape operator of ι . A real hypersurface of $M_n(c)$ is said to be a Hopf real hypersurface when its structure vector is principal.

A connected Riemannian manifold is said to be homogeneous if its group of isometries acts transitively on it. In the paper [1], W. Ambrose and I. M. Singer characterized a Riemannian homogeneous manifold by some kind of a tensor field of type $(1, 2)$ which is called a homogeneous structure tensor (for details see §2, Definition 2.1 and Theorem AS). Later F. Tricerri and L. Vanhecke [12] characterized a naturally reductive Riemannian homogeneous manifold by some kind of a tensor field of type $(1, 2)$ which is called a naturally reductive homogeneous structure tensor (for details see §2, Definition 2.3 and Theorem T-V).

A real hypersurface in a complex space form $M_n(c)$ is said to be a homogeneous real hypersurface if it is an orbit of an analytic subgroup of the group of isometries of $M_n(c)$. In a nonflat complex space form homogeneous real hypersurfaces are all classified (c.f.[10], [3]).

The second author [7] constructed a naturally reductive homogeneous structure tensor T^A on a homogeneous real hypersurface of type (A) in a nonflat complex space form (for details see §2, Theorem NA). Further the second author [8] constructed a homogeneous structure tensor T^B on a homogeneous real hypersurface of type (B) in a nonflat complex space form (for details see §2, Theorem NB).

In this paper we give some characterization theorems of a homogeneous real hypersurface of type (A) in a nonflat complex space form $M_n(c)$ by the Lie derivatives of T^A and T^B in the direction of the structure vector field ξ . Our theorems are:

Theorem 3.1 *Let M be a Hopf real hypersurface in a nonflat complex space form $M_n(c)$ ($c \neq 0$). Then, the Lie derivative $\mathcal{L}_\xi T^A$ of the homogeneous structure T^A in the direction of the structure vector field ξ satisfies the*

following equation:

$$\begin{aligned}
(\mathcal{L}_\xi T^A)_X Y &= -\eta(X)(-\alpha AY - \phi A \phi AY - \frac{c}{4}Y) \\
&\quad + \eta(Y)(-\alpha AX - \phi A \phi AX - \frac{c}{4}X) \\
&\quad - g(A^2 X - \alpha AX - \frac{c}{4}X + \frac{c}{4}\eta(X)\xi, Y)\xi, \quad X, Y \in TM.
\end{aligned} \tag{3.1}$$

Here TM denotes the tangent bundle of M .

Theorem 3.2 *Let M be a Hopf real hypersurface in a nonflat complex space form $M_n(c)$ ($c \neq 0$). If $\mathcal{L}_\xi T^A$ vanishes on M , then M is locally congruent to a homogeneous real hypersurface of type (A) in $M_n(c)$.*

Theorem 3.3 *Let M be a Hopf real hypersurface in a nonflat complex space form $M_n(c)$ ($c \neq 0$). Then, the Lie derivative $\mathcal{L}_\xi T^B$ of the homogeneous structure T^B in the direction of the structure vector field ξ satisfies the following equation:*

$$\begin{aligned}
(\mathcal{L}_\xi T^B)_X Y &= -\frac{\alpha}{2}\eta(X)(\phi A \phi Y + AY) \\
&\quad + \eta(Y)(\phi A \phi AX + \alpha AX + \frac{c}{4}X) \\
&\quad - g(A^2 X - \alpha AX - \frac{c}{4}X, Y)\xi + \frac{3}{2}\alpha^2\eta(X)\eta(Y)\xi, \quad X, Y \in TM.
\end{aligned} \tag{3.11}$$

Theorem 3.4 *Let M be a Hopf real hypersurface in a nonflat complex space form $M_n(c)$ ($c \neq 0$). If $\mathcal{L}_\xi T^B$ vanishes on M , then M is locally congruent to a homogeneous real hypersurface of type (A) in $M_n(c)$.*

2. Preliminaries

In this section we explain preliminary results concerning Riemannian homogeneous structures and real hypersurfaces of a complex space form.

First, we recall a criterion for homogeneity of a Riemannian manifold obtained by W. Ambrose and I. M. Singer [1]. We start with

Definition 2.1. A connected Riemannian manifold (M, g) is said to be homogeneous if the group $I(M)$ of isometries of M acts transitively on M .

On the other hand, local homogeneity is defined by

Definition 2.2. A connected Riemannian manifold (M, g) is said to be locally homogeneous if, for each $p, q \in M$, there exists a neighborhood U of p , a neighborhood V of q and a local isometry $\phi : U \rightarrow V$ such that $\phi(p) = q$.

In the paper [1], Ambrose and Singer gave a criterion for homogeneity of a Riemannian manifold:

Theorem AS([1]). A connected, complete and simply connected Riemannian manifold M is homogeneous if and only if there exists a tensor field T of type $(1, 2)$ on M such that

- (i) $g(T_X Y, Z) + g(Y, T_X Z) = 0$,
- (ii) $(\nabla_X R)(Y, Z) = [T_X, R(Y, Z)] - R(T_X Y, Z) - R(Y, T_X Z)$,
- (iii) $(\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}$,

for $X, Y, Z \in \mathfrak{X}(M)$. Here ∇ denotes the Levi-Civita connection of M , R is the Riemannian curvature tensor of M and $\mathfrak{X}(M)$ is the Lie algebra of all C^∞ vector fields over M .

Furthermore, without the topological conditions of completeness and simply connectedness, the three conditions (i)–(iii) give a criterion for local homogeneity of M .

Remark 2.1. If we put $\tilde{\nabla} := \nabla - T$, then the conditions (i), (ii) and (iii) are equivalent to $\tilde{\nabla}g = 0$, $\tilde{\nabla}R = 0$ and $\tilde{\nabla}T = 0$, respectively.

Next, we present the definition of a naturally reductive homogeneous Riemannian manifold.

Definition 2.3. Let M be a homogeneous Riemannian manifold and g its metric tensor. Then (M, g) is said to be a naturally reductive homogeneous Riemannian manifold if there exists a homogeneous representation

$M = G/K$ with a transitive Lie group G of isometries of M and the isotropy group K of some point $p \in M$ such that for the Lie algebra \mathfrak{g} of G and \mathfrak{k} of K there exists a vector subspace \mathfrak{m} of \mathfrak{g} satisfying

- (i) $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$,
- (ii) $\text{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$,
- (iii) $\langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle Y, [X, Z]_{\mathfrak{m}} \rangle = 0, \quad X, Y, Z \in \mathfrak{m}$,

where $\langle \cdot, \cdot \rangle$ denotes the inner product of \mathfrak{m} induced on \mathfrak{m} from g by identification of \mathfrak{m} with $T_p M$.

If (M, g) is naturally reductive, both the Riemannian metric tensor g and the Riemannian curvature tensor R of M are parallel with respect to the canonical connection $\tilde{\nabla}$ corresponding to the decomposition (i) in Definition 2.3 (cf. [12] p57). Further, for the tensor $T = \nabla - \tilde{\nabla}$, $\tilde{\nabla}_X T = 0$ and $T_X X = 0$ are satisfied for any $X \in TM$. We know the following criterion:

Theorem T-V ([12] p57) *A connected, complete and simply connected Riemannian manifold M is naturally reductive homogeneous if and only if there exists a tensor field T of type (1, 2) on M such that*

- (i) $\tilde{\nabla} g = 0$, (ii) $\tilde{\nabla} R = 0$, (iii) $\tilde{\nabla} T = 0$, (iv) $T_X X = 0$

for $X \in TM$, where $\tilde{\nabla}$ denotes the connection defined by $\tilde{\nabla} = \nabla - T$.

Next, we mention some preliminary results concerning real hypersurfaces. Let $M_n(c)$ ($c \neq 0$) be an n -dimensional complex space form with constant holomorphic sectional curvature c and let g and \tilde{J} be its metric tensor and complex structure, respectively. The standard models for such spaces are the complex projective space $\mathbb{C}P_n(c)$ (for $c > 0$) and the complex hyperbolic space $\mathbb{C}H_n(c)$ (for $c < 0$).

Let M be a real hypersurface of $M_n(c)$. We also denote by g the induced Riemannian metric on M and by ν a local unit normal vector field along M in $M_n(c)$.

The Gauss and Weingarten formulas are:

$$\bar{\nabla}_X \iota_* Y = \iota_* \nabla_X Y + g(AX, Y)\nu, \quad (2.1)$$

$$\bar{\nabla}_X \nu = -\iota_* AX, \quad (2.2)$$

where $\bar{\nabla}$ and ∇ denote the Levi-Civita connection of $M_n(c)$ and M , respectively and A is the shape operator of M in $M_n(c)$.

We define an almost contact metric structure (ϕ, ξ, η, g) on M as usual. That is defined by

$$\iota_*\xi = -\tilde{J}\nu, \quad \eta(X) = g(X, \xi), \quad \iota_*\phi X = (\tilde{J}X)^T, \quad X \in TM, \quad (2.3)$$

where TM denotes the tangent bundle of M and $(\)^T$ the tangential component of a vector. These structure tensors satisfy the following equations:

$$\begin{aligned} \phi^2 &= -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in TM. \end{aligned} \quad (2.4)$$

where I denotes the identity mapping of TM .

From (2.1) and (2.3), we easily have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad (2.5)$$

$$\nabla_X \xi = \phi AX, \quad (2.6)$$

for tangent vectors $X, Y \in TM$.

In our case the Gauss and Coddazi equations of M become

$$\begin{aligned} R(X, Y)Z &= \frac{c}{4} \{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X \\ &\quad - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &\quad + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \quad (2.7)$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4} \{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}. \quad (2.8)$$

A real hypersurface M of $M_n(c)$ is said to be a homogeneous real hypersurface if it is an orbit of an analytic subgroup of the isometry group of $M_n(c)$. We know the complete classification of homogeneous real hypersurfaces of $\mathbb{C}P_n$:

Theorem T([10]). Let M be a homogeneous real hypersurface of $\mathbb{C}P_n$. Then M is locally congruent to one of the following spaces:

- (A₁) a geodesic hypersphere;
- (A₂) a tube over a totally geodesic $\mathbb{C}P_k$ ($1 \leq k \leq n-2$);

- (B) a tube over a complex quadric Q_{n-1} ;
- (C) a tube over $\mathbb{C}P_1 \times \mathbb{C}P_{\frac{n-1}{2}}$ and $n(\geq 5)$ is odd;
- (D) a tube over a complex Grassmann $G_{2,5}$ and $n = 9$;
- (E) a tube over a Hermitian symmetric space $SO(10)/U(5)$ and $n = 15$.

For homogeneity of a real hypersurface in $\mathbb{C}P_n$, there is a criterion obtained by M. Kimura [4]. His theorem is

Theorem K([4]). Let M be an connected real hypersurface in $\mathbb{C}P_n$. Then M has constant principal curvatures and the structure vector ξ is principal if and only if M is congruent to an open subset of a homogeneous real hypersurface.

In $\mathbb{C}H_n$ Berndt [2] obtains the complete classification of Hopf hypersurfaces with constant principal curvatures. His theorem is the following:

Theorem B([2]). Let M be a connected real hypersurface of $\mathbb{C}H_n$ ($n \geq 2$) with constant principal curvatures. Further, assume that the structure vector ξ is principal. Then M is orientable and holomorphic congruent to an open part of one of the following hypersurfaces:

- (A₀) a horosphere in $\mathbb{C}H_n$;
- (A) a tube of radius $r \in \mathbb{R}_+$ over a totally geodesic $\mathbb{C}H_k$ ($0 \leq k \leq n-1$);
- (B) a tube of radius $r \in \mathbb{R}_+$ over a totally geodesic totally real submanifold $\mathbb{R}H_n$.

Here $\mathbb{C}H_0$ means a single point.

For the principal curvatures $\alpha, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and their multiplicities $m_\alpha, m_{\lambda_1}, m_{\lambda_2}, m_{\lambda_3}, m_{\lambda_4}$, we have the following table, where α is the principal curvature corresponding to the principal direction ξ (see [11]):

type	principal curvatures	multiplicities
(A_1)	$\alpha = \sqrt{c} \cot \sqrt{c}r$ $\lambda_1 = \frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}}{2}r$	$m_\alpha = 1$ $m_{\lambda_1} = 2(n-1)$
(A_2)	$\alpha = \sqrt{c} \cot \sqrt{c}r$ $\lambda_1 = \frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}}{2}r$ $\lambda_2 = -\frac{\sqrt{c}}{2} \tan \frac{\sqrt{c}}{2}r$	$m_\alpha = 1$ $m_{\lambda_1} = 2(n-k-1)$ $m_{\lambda_2} = 2k$
(B)	$\alpha = \sqrt{c} \cot \sqrt{c}r$ $\lambda_1 = \frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}}{2}(r - \frac{\pi}{2\sqrt{c}})$ $\lambda_2 = -\frac{\sqrt{c}}{2} \tan \frac{\sqrt{c}}{2}(r - \frac{\pi}{2\sqrt{c}})$	$m_\alpha = 1$ $m_{\lambda_1} = n-2$ $m_{\lambda_2} = n-2$
(C)	$\alpha = \sqrt{c} \cot \sqrt{c}r$ $\lambda_i = \frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}}{2}(r - \frac{\pi i}{2\sqrt{c}})$ $(i = 1, 2, 3, 4)$	$m_\alpha = 1$ $m_{\lambda_i} = n-3 \ (i = 2, 4)$ $m_{\lambda_i} = 2 \ (i = 1, 3)$
(D)	$\alpha = \sqrt{c} \cot \sqrt{c}r$ $\lambda_i = \frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}}{2}(r - \frac{\pi i}{2\sqrt{c}})$ $(i = 1, 2, 3, 4)$	$m_\alpha = 1$ $m_{\lambda_i} = 4 \ (i = 1, 2, 3, 4)$
(E)	$\alpha = \sqrt{c} \cot \sqrt{c}r$ $\lambda_i = \frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}}{2}(r - \frac{\pi i}{2\sqrt{c}})$ $(i = 1, 2, 3, 4)$	$m_\alpha = 1$ $m_{\lambda_i} = 8 \ (i = 2, 4)$ $m_{\lambda_i} = 6 \ (i = 1, 3)$

Table 1: principal curvatures in $\mathbb{C}P_n$

Concerning the principal curvatures α , λ_1 , λ_2 and their multiplicities m_α , m_{λ_1} , m_{λ_2} , we have the following (see [6]):

type	principal curvatures	multiplicities
(A_0)	$\alpha = \sqrt{-c}$ $\lambda_1 = \frac{\sqrt{-c}}{2}$	1 $2n-2$
(A)	$\alpha = \sqrt{-c} \coth \sqrt{-c}r$ $\lambda_1 = \frac{\sqrt{-c}}{2} \coth \frac{\sqrt{-c}}{2}r$ $\lambda_2 = \frac{\sqrt{-c}}{2} \tanh \frac{\sqrt{-c}}{2}r$	1 $2(n-k-1)$ $2k$
(B)	$\alpha = \sqrt{-c} \tanh \sqrt{-c}r$ $\lambda_1 = \frac{\sqrt{-c}}{2} \coth \frac{\sqrt{-c}}{2}r$ $\lambda_2 = \frac{\sqrt{-c}}{2} \tanh \frac{\sqrt{-c}}{2}r$	1 $n-1$ $n-1$

Table 2: principal curvatures in $\mathbb{C}H_n$

For Hopf real hypersurfaces we know the following:

Theorem MO([9], [5]). *Let M be a real hypersurface in $M_n(c)$ whose structure vector ξ is principal with principal curvature α . Then α is a locally constant function. Furthermore, for any principal curvature vector $X \perp \xi$ with $AX = \lambda X$, we get the following equation:*

$$(2\lambda - \alpha)A\phi X = (\alpha\lambda + \frac{c}{2})\phi X. \quad (2.9)$$

Using the table 1 and the table 2, we easily have the following:

Proposition 2.1. *For a real hypersurface of type (A) in $M_n(c)$, we have*

$$\phi A = A\phi, \quad (2.10)$$

$$A^2 - \alpha A - \frac{c}{4}I = -\frac{c}{4}\eta \otimes \xi. \quad (2.11)$$

Here I denotes the identity map on TM .

Proposition 2.2. *For a real hypersurface of type (B) in $M_n(c)$, we have*

$$\phi A + A\phi = -\frac{c}{\alpha}\phi, \quad (2.12)$$

$$A^2 + \frac{c}{\alpha}A - \frac{c}{4}I = (\alpha^2 + \frac{3}{4}c)\eta \otimes \xi. \quad (2.13)$$

In [7] and [8] the second author proved the following theorems:

Theorem NA([7]) *Let M be a homogeneous real hypersurface of type (A) in a nonflat complex space form $M_n(c)$ ($c \neq 0$). Then*

$$T_X^A Y = \eta(Y)\phi AX - \eta(X)\phi AY - g(\phi AX, Y)\xi, \quad X, Y \in TM \quad (2.14)$$

defines a naturally reductive homogeneous structure on M .

Theorem NB([8]) *Let M be a homogeneous real hypersurface of type (B) in a nonflat complex space form $M_n(c)$ ($c \neq 0$). Then*

$$T_X^B Y = \frac{\alpha}{2}\eta(X)\phi Y + \eta(Y)\phi AX - g(\phi AX, Y)\xi. \quad X, Y \in TM \quad (2.15)$$

defines a homogeneous structure on M .

3. Proof of theorems

In this section we shall prove our main theorems.

Firstly, we prove the following theorem:

Theorem 3.1. *Let M be a Hopf real hypersurface in a nonflat complex space form $M_n(c)$ ($c \neq 0$). Then, the Lie derivative $\mathcal{L}_\xi T^A$ of the homogeneous structure T^A in the direction of the structure vector field ξ satisfies the following equation:*

$$\begin{aligned} (\mathcal{L}_\xi T^A)_X Y &= -\eta(X)(-\alpha AY - \phi A \phi AY - \frac{c}{4}Y) \\ &\quad + \eta(Y)(-\alpha AX - \phi A \phi AX - \frac{c}{4}X) \\ &\quad - g(A^2 X - \alpha AX - \frac{c}{4}X + \frac{c}{4}\eta(X)\xi, Y)\xi, \quad X, Y \in TM. \end{aligned} \quad (3.1)$$

Proof. We calculate the Lie derivative $\mathcal{L}_\xi T^A$ by

$$(\mathcal{L}_\xi T^A)_X Y = \mathcal{L}_\xi(T_X^A Y) - T_{\mathcal{L}_\xi X}^A Y - T_X^A(\mathcal{L}_\xi Y), \quad X, Y \in TM.$$

From (2.4), (2.6) and (2.14), we have

$$\begin{aligned} \mathcal{L}_\xi(T_X^A Y) &= g(\nabla_\xi Y, \xi)\phi AX + \eta(Y)\{\nabla_\xi(\phi AX) - \nabla_{\phi AX}\xi\} \\ &\quad - g(\nabla_\xi X, \xi)\phi AY - \eta(X)\{\nabla_\xi(\phi AY) - \nabla_{\phi AY}\xi\} \\ &\quad - g(\nabla_\xi(\phi AX), Y)\xi - g(\phi AX, \nabla_\xi Y)\xi, \end{aligned} \quad (3.2)$$

$$\begin{aligned} T_{\mathcal{L}_\xi X}^A Y &= \eta(Y)\phi A \nabla_\xi X - \eta(Y)\phi A \phi AX - g(\nabla_\xi X, \xi)\phi AY \\ &\quad - g(\phi A \nabla_\xi X, Y)\xi + g(\phi A \phi AX, Y)\xi, \end{aligned} \quad (3.3)$$

$$\begin{aligned} T_X^A(\mathcal{L}_\xi Y) &= g(\nabla_\xi Y, \xi)\phi AX - \eta(X)\phi A(\nabla_\xi Y - \nabla_Y \xi) \\ &\quad - g(\phi AX, \nabla_\xi Y)\xi + g(A^2 X, Y)\xi - \alpha^2 \eta(X)\eta(Y)\xi. \end{aligned} \quad (3.4)$$

From (2.5), we have

$$(\nabla_\xi \phi)X = 0, \quad X \in TM. \quad (3.5)$$

According to the Codazzi equation (2.8) and the equation (2.6), we have

$$(\nabla_\xi A)X = \alpha \phi AX - A \phi AX + \frac{c}{4}\phi X. \quad (3.6)$$

Combining (3.2), (3.3), (3.4), (3.5) with (3.6), we are led to

$$\begin{aligned} (\mathcal{L}_\xi T^A)_X Y &= -\eta(X)(-\alpha AY - \phi A \phi AY - \frac{c}{4}Y) \\ &\quad + \eta(Y)(-\alpha AX - \phi A \phi AX - \frac{c}{4}X) \\ &\quad - g(A^2 X - \alpha AX - \frac{c}{4}X + \frac{c}{4}\eta(X)\xi, Y)\xi. \end{aligned}$$

This proves the theorem. \square

Secondly, we prove the following theorem:

Theorem 3.2. *Let M be a Hopf real hypersurface in a nonflat complex space form $M_n(c)$ ($c \neq 0$). If $\mathcal{L}_\xi T^A$ vanishes on M , then M is locally congruent to a homogeneous real hypersurface of type (A) in $M_n(c)$.*

Proof. By our assumption we have

$$\begin{aligned} (\mathcal{L}_\xi T^A)_X Y &= -\eta(X)(-\alpha AY - \phi A \phi AY - \frac{c}{4}Y) \\ &\quad + \eta(Y)(-\alpha AX - \phi A \phi AX - \frac{c}{4}X) \\ &\quad - g(A^2 X - \alpha AX - \frac{c}{4}X + \frac{c}{4}\eta(X)\xi, Y)\xi \\ &= 0. \end{aligned} \tag{3.7}$$

Substituting $X, Y \in \{\xi\}^\perp$, $AX = \lambda X$ into left side of (3.7), we have

$$\lambda^2 - \alpha\lambda - \frac{c}{4} = 0, \tag{3.8}$$

where $\{\xi\}^\perp$ denotes the orthogonal complement of the vector space spanned by ξ in TM .

So, M has at most three distinct constant principal curvatures. According to Theorem K in §2, M must be locally congruent to a real hypersurface either of type (A) or of type (B). But a real hypersurface of type (B) does not satisfy (3.8) (c.f. Table 1 and Table 2). So M must be of type (A)

On the other hand, for a real hypersurface of type (A), we have the following from Proposition 2.1,

$$\begin{aligned} \phi A \phi AX &= \phi^2 A^2 X \\ &= -A^2 X + \alpha^2 \eta(X). \end{aligned} \tag{3.9}$$

Substituting (3.9) into the right side of (3.1), we conclude that

$$\begin{aligned}
(\mathcal{L}_\xi T^A)_X Y &= -\eta(X)(A^2 Y - \alpha A Y - \frac{c}{4} Y) \\
&\quad + \eta(Y)(A^2 X - \alpha A X - \frac{c}{4} X) \\
&\quad - g(A^2 X - \alpha A X - \frac{c}{4} X + \frac{c}{4} \eta(X) \xi, Y) \xi.
\end{aligned} \tag{3.10}$$

According to Proposition 2.1, the right side of (3.10) vanishes. This proves the theorem. \square

Thirdly, we prove the following theorem:

Theorem 3.3. *Let M be a Hopf real hypersurface in a nonflat complex space form $M_n(c)$ ($c \neq 0$). Then, the Lie derivative $\mathcal{L}_\xi T^B$ of the homogeneous structure T^B in the direction of the structure vector field ξ satisfies the following equation:*

$$\begin{aligned}
(\mathcal{L}_\xi T^B)_X Y &= -\frac{\alpha}{2} \eta(X)(\phi A \phi Y + A Y) \\
&\quad + \eta(Y)(\phi A \phi A X + \alpha A X + \frac{c}{4} X) \\
&\quad - g(A^2 X - \alpha A X - \frac{c}{4} X, Y) \xi + \frac{3}{2} \alpha^2 \eta(X) \eta(Y) \xi, \quad X, Y \in TM.
\end{aligned} \tag{3.11}$$

Proof. We calculate the Lie derivative $\mathcal{L}_\xi T^B$ by

$$(\mathcal{L}_\xi T^B)_X Y = \mathcal{L}_\xi(T_X^B Y) - T_{\mathcal{L}_\xi X}^B Y - T_X^B(\mathcal{L}_\xi Y), \quad X, Y \in TM.$$

Using (2.4), (2.5), (2.6) and (2.15), we have

$$\begin{aligned}
\mathcal{L}_\xi(T_X^B Y) &= \frac{\alpha}{2} g(\nabla_\xi X, \xi) \phi Y + \frac{\alpha}{2} \eta(X) \phi \nabla_\xi Y - \frac{\alpha}{2} \eta(X) \phi A \phi Y \\
&\quad + g(\nabla_\xi Y, \xi) \phi A X - \eta(Y)(\alpha A X + \frac{c}{4} X) \\
&\quad - 2\eta(Y) \phi A \phi A X + \eta(Y) \phi A \nabla_\xi X \\
&\quad + g(\alpha A X + \frac{c}{4} X, Y) \xi + g(\phi A \phi A X, Y) \xi \\
&\quad - g(\phi A \nabla_\xi X, Y) \xi - g(\phi A X, \nabla_\xi Y) \xi,
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
T_{\mathcal{L}_\xi X}^B Y &= \frac{\alpha}{2} g(\nabla_\xi X, \xi) \phi Y + \eta(Y) \phi A \nabla_\xi X - \eta(Y) \phi A \phi A X \\
&\quad - g(\phi A \nabla_\xi X, Y) \xi + g(\phi A \phi A X, Y) \xi,
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
T_X^B(\mathcal{L}_\xi Y) &= \frac{\alpha}{2}\eta(X)\phi\nabla_\xi Y + \frac{\alpha}{2}\eta(X)AY - \frac{\alpha^2}{2}\eta(X)\eta(Y)\xi \\
&\quad + g(\nabla_\xi Y, \xi)\phi AX - g(\phi AX, \nabla_\xi Y)\xi \\
&\quad + g(A^2X, Y)\xi - \alpha^2\eta(X)\eta(Y)\xi.
\end{aligned} \tag{3.14}$$

Combining (3.12), (3.13) with (3.14), we have

$$\begin{aligned}
(\mathcal{L}_\xi T^B)_X Y &= -\frac{\alpha}{2}(\phi A\phi Y + AY) \\
&\quad - \eta(Y)(\phi A\phi AX + \alpha AX + \frac{c}{4}X) \\
&\quad - g(A^2X - \alpha AX - \frac{c}{4}X, Y)\xi + \frac{3}{2}\alpha^2\eta(X)\eta(Y)\xi.
\end{aligned}$$

This proves the theorem. \square

Finally, we prove the following theorem:

Theorem 3.4. *Let M be a Hopf real hypersurface in a nonflat complex space form $M_n(c)$ ($c \neq 0$). If $\mathcal{L}_\xi T^B$ vanishes on M , then M is locally congruent to a homogeneous real hypersurface of type (A) in $M_n(c)$.*

Proof. By our assumption we have

$$\begin{aligned}
(\mathcal{L}_\xi T^B)_X Y &= -\frac{\alpha}{2}(\phi A\phi Y + AY) \\
&\quad - \eta(Y)(\phi A\phi AX + \alpha AX + \frac{c}{4}X) \\
&\quad - g(A^2X - \alpha AX - \frac{c}{4}X, Y)\xi + \frac{3}{2}\alpha^2\eta(X)\eta(Y)\xi \\
&= 0.
\end{aligned} \tag{3.15}$$

Substituting $X, Y \in \{\xi\}^\perp$, $AX = \lambda X$ into the left side of (3.15), we have

$$\lambda^2 - \alpha\lambda - \frac{c}{4} = 0. \tag{3.16}$$

So, M has at most three distinct constant principal curvatures. According to Theorem K in §2, M must be locally congruent to a real hypersurface either of type (A) or of type (B). But a real hypersurface of type (B) does not satisfy (3.6) (c.f. Table 1 and Table 2). So, M must be locally congruent to a homogeneous real hypersurface of type (A).

On the other hand, for a real hypersurface of type (A), from (2.4), (2.10) and (2.11), we always have

$$\begin{aligned}
(\mathcal{L}_\xi T^B)_X Y &= -\frac{\alpha}{2}(\phi^2 AY + AY) \\
&\quad - \eta(Y)(\phi^2 A^2 + \alpha AX + \frac{c}{4}X) \\
&\quad - g(A^2 X - \alpha AX - \frac{c}{4}X, Y)\xi + \frac{3}{2}\alpha^2 \eta(X)\eta(Y)\xi \\
&= \eta(Y)(A^2 X - \alpha AX - \frac{c}{4}X) - g(A^2 X - \alpha AX - \frac{c}{4}X, Y)\xi \\
&= -\frac{c}{4}\eta(X)\eta(Y)\xi + \frac{c}{4}\eta(X)\eta(Y)\xi \\
&= 0.
\end{aligned}$$

This proves the theorem. \square

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